A MIXED TWO-DIMENSIONAL STATIONARY HEAT-CONDUCTION PROBLEM FOR A CYLINDER

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We investigate the exact solution to the problem of calculating the stationary thermal field external to a cylinder on a portion of the surface of which the thermal flow density is constant and on the remaining portion of which the temperature is constant.

We consider an infinitely long cylinder of radius a; on a portion of the surface of cylinder, namely, $|\varphi| < \alpha$, the thermal flow density is constant, and on the remaining portion of the surface the temperature is constant.

Upon introducing a dimensionless thermal flow density, we may, with no loss in generality, write the boundary conditions for the problem under consideration in the form

$$\frac{\partial T}{\partial r} = 1 \quad \text{for} \quad r = a; \ |\varphi| < \alpha, \tag{1}$$

$$T = 0 \quad \text{for} \quad r = a; \ |\varphi| > \alpha. \tag{2}$$

Thus the problem reduces to an integration of Laplace's equation subject to the boundary conditions (1) and (2).

To solve this problem we introduce a conformal transformation of the domain $r > a$ with the aid of the function

$$\varsigma = \xi + i\eta = ia\,\frac{z-a}{z+a}, \tag{3}$$

which maps this domain into the upper halfplane of the complex ς plane and, at the same time, maps points of the boundary contour ($z = ae^{i\varphi}$) into points of the real axis, $\mathrm{Re}\,\varsigma = \xi = -a\tan\varphi/2$.

Through this transformation our problem reduces to calculating the stationary thermal field in the halfspace $\eta > 0$, subject to the following boundary conditions on the surface $\eta = 0$:

$$\frac{\partial T}{\partial \eta} = \frac{2a^2}{\xi^2 + a^2} \quad \text{for} \quad |\xi| < a\,\mathrm{tg}\,\frac{\alpha}{2}, \tag{4}$$

$$T = 0 \quad \text{for} \quad |\xi| > a\,\mathrm{tg}\,\frac{\alpha}{2}. \tag{5}$$

Introducing the normalized coordinates

$$\xi_1 = \frac{\xi}{a\,\mathrm{tg}\,\frac{\alpha}{2}} = -\frac{\sin\varphi}{\mathrm{tg}\,\frac{\alpha}{2}}\,\frac{2ar}{r^2 + a^2 + 2ar\cos\varphi},$$

$$\eta_1 = \frac{\eta}{a\,\mathrm{tg}\,\frac{\alpha}{2}} = \mathrm{ctg}\,\frac{\alpha}{2}\,\frac{r^2a^2}{r^2 + a^2 + 2ar\cos\varphi},$$

we may write these conditions in the form:

$$\left.\frac{\partial T}{\partial \eta_1}\right|_{\eta_1=0} = \frac{2a\,\mathrm{tg}\,\frac{\alpha}{2}}{1 + \xi_1^2\mathrm{tg}^2\,\frac{\alpha}{2}} \quad \text{for} \quad |\xi_1| < 1, \tag{6}$$

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$$T|_{\eta_1=0} = 0 \quad \text{for} \quad |\xi_1| > 1. \tag{7}$$

We then determine T in the usual way

$$T = \int_0^\infty A(p) \exp(-p\eta_1) \cos p\xi_1 dp, \tag{8}$$

thus reducing the problem to one of solving the following system of integral equations:

$$\int_0^\infty A(p)\, p \cos p\xi_1 dp = f(\xi_1) \quad \text{for} \quad |\xi_1| < 1, \tag{9}$$

$$\int_0^\infty A(p) \cos p\xi_1 dp = 0 \quad \text{for} \quad |\xi_1| > 1, \tag{10}$$

where

$$f(\xi_1) = -\frac{2a \operatorname{tg} \frac{\alpha}{2}}{1 + \xi_1^2 \operatorname{tg}^2 \frac{\alpha}{2}}.$$

The solution of a system of this form is deduced in [1] and may be expressed by the formula

$$A(p) = \int_0^1 \psi(t) J_0(pt)\, dt, \tag{11}$$

where

$$\psi(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{u\varphi(u)}{\sqrt{t^2 - u^2}} du; \quad \varphi(x) = \int f(x)\, dx.$$

For the form of $f(\xi_1)$ given above, $\varphi(\xi_1) = -2a \arctan(\xi_1 \tan \alpha/2)$, and for the function $\psi(t)$ we find, after integrating and simplifying,

$$\psi(t) = -2a \operatorname{tg} \frac{\alpha}{2} \frac{t}{\sqrt{1 + t^2 \operatorname{tg}^2 \frac{\alpha}{2}}}. \tag{12}$$

Substituting this expression into Eq. (11), and after that, into Eq. (8), and interchanging orders of integration, we obtain

$$T = -2a \operatorname{tg} \frac{\alpha}{2} \int_0^1 \frac{tdt}{\sqrt{1 + t^2 \operatorname{tg}^2 \frac{\alpha}{2}}} \int_0^\infty J_0(pt) \exp(-p\eta_1) \cos p\xi_1 dp$$

$$= -\sqrt{2} a \operatorname{tg} \frac{\alpha}{2} \int_0^1 t \sqrt{\frac{t^2 + \eta_1^2 - \xi_1^2 + \sqrt{(t^2 + \eta_1^2 - \xi_1^2)^2 + 4\xi_1^2 \eta_1^2}}{[(t^2 + \eta_1^2 - \xi_1^2)^2 + 4\xi_1^2 \eta_1^2] \left[1 + t^2 \operatorname{tg}^2 \frac{\alpha}{2}\right]}}\, dt$$

$$= -\sqrt{2} a \ln \frac{\operatorname{tg} \frac{\alpha}{2} \sqrt{\beta_2^2 \operatorname{tg}^2 \frac{\alpha}{2} + 2\delta\beta_2 - \gamma^2 \operatorname{tg}^2 \frac{\alpha}{2}} + \beta_2 \operatorname{tg}^2 \frac{\alpha}{2} + \delta}{\operatorname{tg} \frac{\alpha}{2} \sqrt{\beta_1^2 \operatorname{tg}^2 \frac{\alpha}{2} + 2\delta\beta_1 - \gamma^2 \operatorname{tg}^2 \frac{\alpha}{2}} + \beta_1 \operatorname{tg}^2 \frac{\alpha}{2} + \delta}, \tag{13}$$

where

$$\beta_1 = 2\eta_1^2; \; \beta_2 = 1 + \eta_1^2 - \xi_1^2 + \sqrt{(1 + \eta_1^2 - \xi_1^2)^2 + 4\xi_1^2 \eta_1^2}; \quad \gamma = 2\xi_1\eta_1; \quad \delta = 1 + (\xi_1^2 - \eta_1^2) \operatorname{tg}^2 \frac{\alpha}{2}.$$

From the general formula obtained it follows, in particular, that

$$T|_{\substack{r=a \\ |\varphi|<\alpha}} = -\sqrt{2} a \ln \left[\frac{2 \sec \frac{\alpha}{2} \sqrt{\operatorname{tg}^2 \frac{\alpha}{2} - \operatorname{tg}^2 \frac{\varphi}{2} + 2\left(\operatorname{tg}^2 \frac{\alpha}{2} - \operatorname{tg}^2 \frac{\varphi}{2}\right)}}{1 + \operatorname{tg}^2 \frac{\varphi}{2}} + 1 \right]. \tag{14}$$

From this we see that when $\varphi \to \alpha$ (i.e., when $\eta_1 = 0$; $\xi_1 \to 1$) the value of $T|_{r=a}$ tends to zero, i.e., the temperature distribution at the boundary, which we have obtained, goes over continuously into the given temperature.

For other typical particular cases we have

$$T|_{\varphi=0} = -\sqrt{2}a \ \ln \frac{2 \sec \frac{\alpha}{2} \sqrt{\mathrm{tg}^2 \frac{\alpha}{2} + \left(\frac{r-a}{r+a}\right)^2} + 2\mathrm{tg}^2 \frac{\alpha}{2} + \left(\frac{r-a}{r+a}\right)^2 + 1}{\frac{r-a}{r+a}\left(2 + \frac{r-a}{r+a}\right) + 1}, \tag{15}$$

$$T|_{\varphi=\pi} = -\sqrt{2}a \ \ln \frac{2 \sec \frac{\alpha}{2} \sqrt{\mathrm{tg}^2 \frac{\alpha}{2} + \left(\frac{r+a}{r-a}\right)^2} + 2\mathrm{tg}^2 \frac{\alpha}{2} + \left(\frac{r+a}{r-a}\right)^2 + 1}{\frac{r+a}{r-a}\left(2 + \frac{r+a}{r-a}\right) + 1}, \tag{16}$$

$$T_{r\to\infty} = 2\sqrt{2}a \ln \cos \frac{\alpha}{2}. \tag{17}$$

Thus we have a complete solution of our problem in closed form.

We note, in conclusion, that in accordance with the method of successive approximations presented in [1], the expression that we have obtained for T is a first approximation for the calculation of convective heat-transfer conditions on the surface $r = a$; $|\varphi| < \alpha$. Moreover, by virtue of known analogies, the solution we have obtained can be used to calculate an arbitrary stationary potential field external to a cylinder subject to corresponding boundary conditions on its surface.

NOTATION

T	is the reduced temperature;
$\partial T/\partial r|_{r=a}$	is the dimensionless heat-flux density on the cylinder surface;
r, φ	are the polar coordinates;
a, α	are the parameters of design model;
z, ζ	are the complex variables;
ξ, η	are the Cartesian coordinates in the reflected plane;
ξ_1, η_1	are the fixed coordinates;
u, x	are the variables of integration;
$A(p), \psi(t), \varphi(u), f(\xi)$	are the auxiliary functions;
$\beta_1, \beta_2, \gamma, \delta$	are the parameters;
J_0	are the cylindrical functions.

LITERATURE CITED

1. Yu. Ya. Iossel' and R. A. Pavlovskii, Inzh.-Fiz. Zh., 19, No. 4 (1970).